

# Any component of moduli of polarized hyperkähler manifolds is dense in its deformation space

Sasha Anan'in, Misha Verbitsky<sup>1</sup>

## Abstract

Let  $M$  be a compact hyperkähler manifold, and  $W$  the coarse moduli of complex deformations of  $M$ . Every positive integer class  $v$  in  $H^2(M)$  defines a divisor  $D_v$  in  $W$  consisting of all algebraic manifolds polarized by  $v$ . We prove that every connected component of this divisor is dense in  $W$ .

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## 1 Introduction

### 1.1 Hyperkähler manifolds and moduli spaces

Throughout this paper, a **hyperkähler manifold** means a “compact complex manifold admitting a Kähler structure and a holomorphically symplectic form.” A hyperkähler manifold  $M$  is called **simple** if  $\pi_1(M) = 0$  and  $H^{2,0}(M) = \mathbb{C}$ . By Bogomolov's theorem (see [Bes] and [Bo1]), any hyperkähler manifold has a finite covering which is a product of simple hyperkähler manifolds and compact tori. Throughout this paper, we shall silently assume that all our hyperkähler manifolds are simple. The results

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that we prove can be stated and proven for general hyperkähler manifolds, but to do so would destroy the clarity of the exposition.

For a background story on hyperkähler manifolds, their construction, and properties, please see [Bea] and [Bes]. The moduli spaces of hyperkähler manifolds are discussed at great length in [V2].

The moduli space of complex structures on a given smooth oriented manifold  $M$  is defined, following Kodaira and Spencer, as the quotient of the Fréchet manifold of all integrable almost complex structures  $\text{Comp}$  by the action of the group of orientation-preserving diffeomorphisms  $\text{Diff}^+$ , which is considered as a Fréchet Lie group. We denote by  $\text{Comp}_0 \subset \text{Comp}$  the open set consisting of all complex structures on  $M$  admitting a compatible Kähler metric and a compatible holomorphically symplectic structure. The quotient  $\text{Mod} := \text{Comp}_0 / \text{Diff}^+$  is called a **coarse moduli space** of hyperkähler manifolds. It is a complex analytic space, usually non-Hausdorff.

It is well known that a generic point  $I \in \text{Mod}$  corresponds to a non-algebraic complex structure on  $M$ . In fact, the manifold  $(M, I)$  has no divisors, because the corresponding Neron-Severi group  $H^{1,1}(M, \mathbb{Z}) := H^{1,1}(M) \cap H^2(M, \mathbb{Z})$  is zero (see [F]). The algebraic points of  $\text{Mod}$  sit on a countable union of divisors in  $\text{Mod}$ , which is known to be dense in  $\text{Mod}$  ([F], [V0]).

In this paper we prove that each of these divisors is itself dense in  $\text{Mod}$ . This result is known when  $M$  is a K3 surface (this follows from a statement known as “Eichler Criterion”; see Remark 2.4).

## 1.2 Lelong numbers, SYZ conjecture and Gromov’s precompactness theorem

The original motivation for this work came from a research on the so-called hyperkähler SYZ conjecture ([V3]). This conjecture, which is a version of a (more general) abundance conjecture of Kawamata, states that a nef bundle on a hyperkähler manifold is semiample. More specifically, one is interested in holomorphic line bundles  $L$  which are nef, and for which the Bogomolov-Beauville-Fujiki square of  $c_1(L)$  vanishes:  $q(c_1(L), c_1(L)) = 0$  (for a definition of Bogomolov-Beauville-Fujiki form, see Subsection 1.3). Such line bundles are called **parabolic**. Any nef bundle admits a singular metric with semipositive curvature (this follows from general results on weak compactness of positive currents). If this metric is not “very singular”,  $L$  is effective ([V3], [V4]). The “not very singular” above refers to the vanishing of the so-called Lelong numbers of the curvature current; these numbers, defined for positive closed  $(p, p)$ -currents, vanish for all smooth currents, and measure the geometric “strength” of its singularities in the general case, taking values in  $\mathbb{R}^{\geq 0}$ .

The Lelong numbers are known to be upper semicontinuous in the cur-

rent topology. This means, in particular, that any cohomology class  $\eta$  which is represented as a limit of currents with Lelong numbers bounded from below would have positive Lelong numbers.

Suppose now that  $\eta \in H^{1,1}(M, \mathbb{R})$  is a nef class on a non-algebraic hyperkähler manifold satisfying  $q(\eta, \eta) = 0$  (such class is also called **parabolic**). It is proven in [V4] that the Lelong sets (sets where the Lelong numbers are bounded from below by a positive number) of  $\eta$  are coisotropic with respect to the holomorphic symplectic structure. However, all complex subvarieties of a generic non-algebraic hyperkähler manifold are hyperkähler ([V1]), hence they cannot be coisotropic. This means that any parabolic nef current on a generic non-algebraic manifold has vanishing Lelong numbers.

To apply this argument, we need to approximate a given non-algebraic manifold with a nef current by a sequence of algebraic manifolds with a rational parabolic current, in a controlled way. To keep this approximation controlled, the manifolds should belong to the same algebraic family.

Generally speaking, such a sequence is hard to produce. For a K3 such approximations are well known, and much used since the earliest works on K3 in the 1960-es. It is known, in particular, that the variety of quartic surfaces is dense in the moduli of all (non-algebraic) K3 surfaces.

In this paper, we generalize this theorem, proving that the moduli of polarized hyperkähler manifolds is a dense subset in the moduli of all (non-algebraic) deformations. More precisely, given a rational cohomology class  $\eta \in H^2(M, \mathbb{Q})$ , satisfying  $q(\eta, \eta) > 0$ , we show that any given  $M$  can be approximated by deformations of  $M$  which satisfy  $\eta \in H^{1,1}(M_1)$ .

It is interesting that even this result seems to be quite hard to prove. Our proof relies on rationality of  $\eta$  and does not work when  $\eta$  is irrational, or  $q(\eta, \eta) \leq 0$ , though the statement is most likely true in this case as well.

Another interesting subject connected to the present problem is the Gromov precompactness theorem. Consider the Gromov-Hausdorff distance, which is a metric on the set of all compact metric spaces. Gromov has shown that the set  $\mathfrak{S}$  of all Riemannian manifolds of semipositive Ricci curvature and bounded diameter is *precompact*, which means that its completion with respect to the Gromov-Hausdorff distance is compact. This result highlights the notion of *Gromov-Hausdorff limits*, that is, the metric spaces appearing as limits of some family of Riemannian manifolds with respect to this distance.

For a finite-dimensional set of Ricci-flat manifolds, the Gromov-Hausdorff limits are especially interesting because of the Gromov precompactness theorem.

Let now  $W_\eta$  be the moduli of polarized hyperkähler manifolds, with  $\eta \in H^2(M)$  being a polarization. The space  $W_\eta$  is known to be quasiprojective (see [Vi]); moreover, it is locally symmetric ([GHS1]). To compactify  $W_\eta$ ,

one usually uses the so-called Baily-Borel compactification. However, points of  $W_\eta$  correspond to polarized hyperkähler manifolds, which are equipped with a canonical Ricci-flat Kähler metric in the same cohomology class as  $\eta$  ([Bes]). This allows one to define the Gromov-Hausdorff compactification of the family associated with  $W_\eta$ . It is unknown (except for the K3, where some partial results are obtained) how the Gromov-Hausdorff completion of  $W_\eta$  corresponds to the Baily-Borel one.

However, the most interesting limits occur when one varies the Kähler class of a Ricci-flat Kähler metric on  $M$ . For instance, when one takes a limit of Ricci-flat Kähler metrics with a Kähler classes  $\omega_i$  converging to  $\eta$  with  $q(\eta, \eta) = 0$ . The hyperkähler SYZ conjecture predicts that the Gromov-Hausdorff limit of the corresponding Ricci-flat metrics would give  $\mathbb{C}P^n$ ,  $n = \frac{1}{2} \dim_{\mathbb{C}} M$  ([KS]).

Arguing this way, one would necessarily come to study the set of all Ricci-flat Kähler metric on  $M$  and their Gromov-Hausdorff limits. However, for the whole moduli space  $W$  of deformations there is no analogue of Baily-Borel compactification, hence the Gromov-Hausdorff compactification has no obvious algebraic counterpart.

In the present paper we show that  $W_\eta$  is dense in  $W$ , for rational  $\eta$ . Together with the known compactification results for  $W_\eta$ , this result could lead, at least in theory, to a better understanding of the Gromov-Hausdorff compactification of the space of all hyperkähler metrics.

### 1.3 Bogomolov-Beauville-Fujiki form and the mapping class group

For a better understanding of the moduli space geometry, some basic facts about topology of hyperkähler manifolds should be stated. We follow [V2].

Let  $\Omega$  be a holomorphic symplectic form on a hyperkähler manifold  $M$ . Bogomolov [Bo2] and Beauville [Bea] defined the following bilinear symmetric 2-form on  $H^2(M)$  :

$$\begin{aligned} \tilde{q}(\eta, \eta') := & \int_M \eta \wedge \eta' \wedge \Omega^{n-1} \wedge \overline{\Omega}^{n-1} - \\ & - \frac{(n-1)}{n} \frac{\left( \int_M \eta \wedge \Omega^{n-1} \wedge \overline{\Omega}^n \right) \cdot \left( \int_M \eta' \wedge \Omega^n \wedge \overline{\Omega}^{n-1} \right)}{\int_M \Omega^n \wedge \overline{\Omega}^n}, \end{aligned} \quad (1.1)$$

where  $4n = \dim_{\mathbb{R}} M$ .

The form  $\tilde{q}$  is topological by its nature.

**Theorem 1.1** [F]: *Let  $M$  be a simple hyperkähler manifold of real dimension  $4n$ . Then there exist a bilinear, symmetric, primitive non-degenerate integer 2-form  $q : H^2(M, \mathbb{Z}) \otimes H^2(M, \mathbb{Z}) \rightarrow \mathbb{Z}$  and a positive constant  $c \in \mathbb{Z}$  such that  $\int_M \eta^{2n} = cq(\eta, \eta)^n$  for all  $\eta \in H^2(M)$ . Moreover,  $q$  is proportional to the form  $\tilde{q}$  of (1.1), and has signature  $(3, b_2 - 3)$  (with 3 pluses and  $b_2 - 3$  minuses). ■*

Let  $\text{Diff}^+$  denote the group of orientation-preserving diffeomorphisms of  $M$ , and  $\text{Diff}^0$  its connected component, also known as a group of isotopies. The quotient group  $\Gamma := \text{Diff}^+ / \text{Diff}^0$  is called the **mapping class group** of  $M$ . In [V2] it was shown that  $\Gamma$  preserves the Bogomolov-Beauville-Fujiki form on  $H^2(M)$  and that the corresponding homomorphism to the orthogonal group  $\Gamma \rightarrow \text{O}(H^2(M), q)$  has finite kernel. It was also shown that the image of  $\Gamma$  in  $\text{O}(H^2(M), q)$  is commensurable to the group  $\text{O}(H^2(M, \mathbb{Z}), q)$  of isometries of the integer lattice.

#### 1.4 Teichmüller space and the moduli space

To state our main result in precise terms, we have to give a more explicit description of the moduli space of a hyperkähler manifold. We follow [V2].

Let  $M$  be a hyperkähler manifold (compact and simple, as usual), and  $\text{Comp}_0$  the Fréchet manifold of all complex structures of hyperkähler type on  $M$ . The quotient  $\text{Teich} := \text{Comp}_0 / \text{Diff}^0$  of  $\text{Comp}_0$  by isotopies is a finite-dimensional complex analytic space by the same Kodaira-Spencer arguments as used to show that  $\text{Mod} = \text{Comp} / \text{Diff}^+$  is complex analytic, where  $\text{Comp}$  is a Fréchet manifold of all integrable complex, oriented structures on  $M$ . This quotient is called a **Teichmüller space** of  $M$ . When  $M$  is a complex curve, the quotient  $\text{Comp} / \text{Diff}^0$  is the Teichmüller space of this curve.

The mapping class group  $\Gamma = \text{Diff}^+ / \text{Diff}^0$  acts on  $\text{Teich}$  in a usual way, and its quotient is the moduli space of  $M$ .

As shown in [H2],  $\text{Teich}$  has a finite number of connected components. Take a connected component  $\text{Teich}^I$  containing a given complex structure  $I$ , and let  $\Gamma^I \subset \Gamma$  be the set of elements of  $\Gamma$  fixing this component. Since  $\text{Teich}$  has only a finite number of connected components,  $\Gamma^I$  has finite index in  $\Gamma$ . On the other hand, as shown in [V2], the image of the group  $\Gamma$  is commensurable to  $\text{O}(H^2(M, \mathbb{Z}), q)$ .

In [V2, Lemma 2.6] it was proved that any hyperkähler structure on a given simple hyperkähler manifold is also simple. Therefore,  $H^{2,0}(M, I') = \mathbb{C}$  for all  $I' \in \text{Comp}$ . This observation is a key to the following well-known definition.

**Definition 1.2:** Let  $(M, I)$  be a hyperkähler manifold, and  $\text{Teich}$  its Teichmüller space. Consider a map  $\text{Per} : \text{Teich} \rightarrow \mathbb{P}H^2(M, \mathbb{C})$ , sending  $J$  to the line  $H^{2,0}(M, J) \in \mathbb{P}H^2(M, \mathbb{C})$ . It is easy to see that  $\text{Per}$  maps  $\text{Teich}$  into the open subset of a quadric, defined by

$$\mathbb{P}\text{er} := \{l \in \mathbb{P}H^2(M, \mathbb{C}) \mid q(l, l) = 0, q(l, \bar{l}) > 0\}.$$

The map  $\text{Per} : \text{Teich} \rightarrow \mathbb{P}\text{er}$  is called the **period map**, and the set  $\mathbb{P}\text{er}$  the **period space**.

The following fundamental theorem is due to F. Bogomolov [Bo2].

**Theorem 1.3:** *Let  $M$  be a simple hyperkähler manifold, and  $\text{Teich}$  its Teichmüller space. Then the period map  $\text{Per} : \text{Teich} \rightarrow \mathbb{P}\text{er}$  is a local diffeomorphism (that is, an étale map). Moreover, it is holomorphic. ■*

**Remark 1.4:** *Bogomolov's theorem implies that  $\text{Teich}$  is smooth. However, it is not necessarily Hausdorff (and it is non-Hausdorff even in the simplest examples). ■*

## 1.5 The polarized Teichmüller space

In [V4, Corollary 2.6], the following proposition was deduced from [Bou] and [DP].

**Theorem 1.5:** *Let  $M$  be a simple hyperkähler manifold, such that all integer  $(1, 1)$ -classes satisfy  $q(\nu, \nu) \geq 0$ . Then its Kähler cone is one of two connected components of the set  $K := \{\nu \in H^{1,1}(M, \mathbb{R}) \mid q(\nu, \nu) > 0\}$ . ■*

Consider an integer vector  $\eta \in H^2(M)$  which is positive, that is, satisfies  $q(\eta, \eta) > 0$ . Denote by  $\text{Teich}^\eta$  the set of all  $I \in \text{Teich}$  such that  $\eta$  is of type  $(1, 1)$  on  $(M, I)$ . The space  $\text{Teich}^\eta$  is a closed divisor in  $\text{Teich}$ . Indeed, by Bogomolov's theorem, the period map  $\text{Per} : \text{Teich} \rightarrow \mathbb{P}\text{er}$  is étale, but the image of  $\text{Teich}^\eta$  is the set of all  $l \in \mathbb{P}\text{er}$  which are orthogonal to  $\eta$ ; this condition defines a closed divisor  $C_\eta$  in  $\mathbb{P}\text{er}$ , hence  $\text{Teich}^\eta = \text{Per}^{-1}(C_\eta)$  is also a closed divisor.

When  $I \in \text{Teich}^\eta$  is generic, Bogomolov's theorem implies that the space of rational  $(1, 1)$ -classes  $H^{1,1}(M, \mathbb{Q})$  is one-dimensional and generated by  $\eta$ . This is seen from the following argument. Locally around a given point  $I$  the period map  $\text{Teich}^\eta \rightarrow \mathbb{P}\text{er}$  is surjective on the set  $\mathbb{P}\text{er}^\eta$  of all  $I \in \mathbb{P}\text{er}$  for which  $\eta \in H^{1,1}(M, I)$ . However, the Hodge-Riemann relations give

$$\mathbb{P}\text{er}^\eta = \{l \in \mathbb{P}\text{er} \mid q(\eta, l) = 0\}. \quad (1.2)$$

Denote the set of such points of  $\text{Teich}^\eta$  by  $\text{Teich}_{\text{gen}}^\eta$ . It follows from Theorem 1.5 that, for any  $I \in \text{Teich}_{\text{gen}}^\eta$ , either  $\eta$  or  $-\eta$  is a Kähler class on  $(M, I)$ .

Consider a connected component  $\text{Teich}^{\eta, I}$  of  $\text{Teich}^\eta$ . Changing the sign of  $\eta$  if necessary, we may assume that  $\eta$  is Kähler on  $(M, I)$ . By Kodaira's theorem about stability of Kähler classes,  $\eta$  is Kähler in some neighbourhood  $U \subset \text{Teich}^{\eta, I}$  of  $I$ . Therefore, the sets

$$V_+ := \{I \in \text{Teich}_{\text{gen}}^\eta \mid \eta \text{ is Kähler on } (M, I)\}$$

and

$$V_- := \{I \in \text{Teich}_{\text{gen}}^\eta \mid -\eta \text{ is Kähler on } (M, I)\}$$

are open in  $\text{Teich}_{\text{gen}}^\eta$ . It is easy to see that  $\text{Teich}_{\text{gen}}^\eta$  is a complement to a union of countably many divisors in  $\text{Teich}^\eta$  corresponding to the points  $I' \in \text{Teich}^\eta$  with  $\text{rk Pic}(M, I') > 1$ . Therefore, for any connected open subset  $U \subset \text{Teich}^\eta$ , the intersection  $U \cap \text{Teich}_{\text{gen}}^\eta$  is connected. Since  $\text{Teich}_{\text{gen}}^\eta$  is represented as a disjoint union of open sets  $V_+ \sqcup V_-$ , every connected component of  $\text{Teich}_{\text{gen}}^\eta$  and of  $\text{Teich}^\eta$  is contained in  $V_+$  or in  $V_-$ . We obtained the following corollary.

**Corollary 1.6:** *Let  $\eta \in H^2(M)$  be a positive integer vector,  $\text{Teich}^\eta$  the corresponding divisor in the Teichmüller space, and  $\text{Teich}^{\eta, I}$  a connected component of  $\text{Teich}^\eta$  containing a complex structure  $I$ . Assume that  $\eta$  is Kähler on  $(M, I)$ . Then  $\eta$  is Kähler for all  $I' \in \text{Teich}^{\eta, I}$  which satisfy  $\text{rk } H^{1,1}(M, \mathbb{Q}) = 1$ . ■*

We call the set  $\text{Teich}_{\text{pol}}^\eta$  of all  $I \in \text{Teich}^\eta$  for which  $\eta$  is Kähler the **polarized Teichmüller space**, and  $\eta$  its **polarization**. From the above arguments it is clear that the polarized Teichmüller space is open and dense in  $\text{Teich}^\eta$ .

The quotient  $\mathcal{M}_\eta$  of  $\text{Teich}_{\text{pol}}^\eta$  by the subgroup of a mapping class group fixing  $\eta$  is called the **moduli of polarized hyperkähler manifolds**. It is known (due to the general theory which goes back to Viehweg and Grothendieck that  $\mathcal{M}_\eta$  is Hausdorff and quasiprojective (see e.g. [Vi] and [GHS1]).

We conclude that there are countably many quasiprojective divisors  $\mathcal{M}_\eta$  immersed in the moduli space  $\text{Mod}$  of hyperkähler manifolds. Moreover, every algebraic complex structure belongs to one of these divisors. However, these divisors need not to be closed. Indeed, as we prove in this paper, each of the  $\mathcal{M}_\eta$  is dense in  $\text{Mod}$ .

The main result of the present paper is the following theorem.

**Theorem 1.7:** *Let  $M$  be a compact, simple hyperkähler manifold,  $\text{Teich}^I$  a connected component of its Teichmüller space, and  $\text{Teich}^I \xrightarrow{\Psi} \text{Teich}^I / \Gamma^I =$*

Mod its projection to the moduli of complex structures. Consider a positive vector  $\eta \in H^2(M, \mathbb{Z})$ , and let  $\text{Teich}^{I, \eta}$  be the corresponding connected component of the polarized Teichmüller space. Assume that  $b_2(M) > 3$ . Then the image  $\Psi(\text{Teich}^{I, \eta})$  is dense in Mod.

We deduce Theorem 1.7 from Proposition 3.2 in Section 2, and prove Proposition 3.2 in Section 3.

**Remark 1.8:** We assumed positivity of  $\eta$  in the statement of Theorem 1.7, but this assumption is completely unnecessary. In fact, for  $\eta$  non-positive, the proof of Theorem 1.7 becomes easier (Remark 3.12). ■

## 2 Torelli theorem and polarizations

In this Section, we reduce Theorem 1.7 to a statement about lattices and arithmetic groups, proven in Section 3.

Let  $M$  be a topological space, not necessarily Hausdorff. We say that points  $x, y \in M$  are **inseparable** (denoted  $x \sim y$ ) if for any open subsets  $U \ni x, V \ni y$ , one has  $U \cap V \neq \emptyset$ .

**Theorem 2.1** [V2, Theorem 1.14, Theorem 1.16]: *Let  $\text{Teich}$  be a Teichmüller space of a hyperkähler manifold, and  $\sim$  the inseparability relation defined above. Then  $\sim$  is an equivalence relation, and the quotient  $\text{Teich}_b := \text{Teich}/\sim$  is a smooth, Hausdorff, complex analytic manifold. Moreover, the period map  $\text{Per} : \text{Teich} \rightarrow \mathbb{P}\text{er}$  induces a complex analytic diffeomorphism  $\text{Teich}_b^I \rightarrow \mathbb{P}\text{er}$  for each connected component  $\text{Teich}_b^I$  of  $\text{Teich}_b$ . ■*

**Remark 2.2:** As shown by Huybrechts [H1], inseparable points on a Teichmüller space correspond to bimeromorphically equivalent hyperkähler manifolds. The Hausdorff quotient  $\text{Teich}_b = \text{Teich}/\sim$  is called the **birational Teichmüller space** of  $M$ . ■

By construction, the action of the mapping class group  $\Gamma$  on  $\text{Teich}_b$  is compatible with the natural action of  $O(H^2(M, \mathbb{Z}), q)$  on  $\mathbb{P}\text{er}$ . Define the **birational moduli space** as  $\text{Mod}_b := \text{Teich}_b/\Gamma$ . The space  $\text{Mod}_b$  is obtained by gluing together some (not all) inseparable points in Mod. By Theorem 2.1,  $\text{Mod}_b = \mathbb{P}\text{er}/\Gamma^I$ , where  $\Gamma^I$  is a subgroup of  $\Gamma$  fixing a connected component  $\text{Teich}^I$  of the Teichmüller space. As follows from [V2, Theorem 3.5] (see also Subsection 1.3), the image of  $\Gamma^I$  in  $\text{Aut}(\mathbb{P}\text{er})$  is a finite index subgroup in  $O(H^2(M, \mathbb{Z}), q)$ .



It is well known that the homogeneous space

$$\mathbb{P}er = \{l \in \mathbb{P}H^2(M, \mathbb{C}) \mid q(l, l) = 0, q(l, \bar{l}) > 0\}$$

is naturally identified with the Grassmanian

$$\mathrm{Gr}^{++}(H^2(M, \mathbb{R})) \cong \mathrm{SO}(3, b_2 - 3) / \mathrm{SO}(2) \times \mathrm{SO}(1, b_2 - 3)$$

of oriented positive 2-dimensional planes in  $H^2(M, \mathbb{R})$ . This identification is performed as follows: to each line  $l \in \mathbb{P}H^2(M, \mathbb{C})$  one associates the plane spanned by  $\mathrm{Re}(l), \mathrm{Im}(l)$ . Under this identification, the image of the polarized Teichmüller space  $\mathrm{Teich}^\eta$  is the space of all 2-dimensional planes  $P \in \mathrm{Gr}^{++}(H^2(M, \mathbb{R}))$  orthogonal to  $\eta$  (see (1.2)). Then Theorem 1.7 is implied by the following statement.

**Theorem 2.3:** *Let  $M$  be a simple, compact hyperkähler manifold,  $V := H^2(M, \mathbb{R})$  its second cohomology,  $L := H^2(M, \mathbb{Z})$ , and  $q$  the Bogomolov-Beauville-Fujiki form on  $V$ . Given a positive integer vector  $\eta \in L$ , denote by  $\mathrm{Gr}^{++}(\eta^\perp) \subset \mathrm{Gr}^{++}(V)$  the space of all planes orthogonal to  $\eta$ . Consider a finite index subgroup  $G \subset \mathrm{SO}(H^2(M, \mathbb{Z}), q)$  acting on  $\mathrm{Gr}^{++}(V)$  in a natural way. Then  $G \cdot \mathrm{Gr}^{++}(\eta^\perp)$  is dense in  $\mathrm{Gr}^{++}(V) = \mathbb{P}er$ .*

Theorem 2.3 is implied by a more general Proposition 3.2 proven in the next section using the framework laid down in [AGr].

**Remark 2.4:** *When  $M$  is a K3 surface, the Bogomolov-Beauville-Fujiki form is unimodular, and the mapping class group is generated by appropriate reflections. From a statement known as “Eichler’s criterion” (see [GHS2, Proposition 3.3(i)]), the mapping class group acts transitively on the set of integer vectors of a given length in  $H^2(M)$ . Theorem 2.3 follows from this observation easily. When the Eichler’s criterion cannot be applied, its proof is more complicated. ■*

### 3 Arithmetic subgroups in $\mathrm{O}(p, q)$

Let  $V$  be a finite-dimensional  $\mathbb{R}$ -vector space equipped with a non-degenerate symmetric form  $\langle \cdot, \cdot \rangle$  and  $W$  an  $\mathbb{R}$ -vector subspace in  $V$ . Denote by  $\mathrm{Gr}_{++}(W)$  (respectively, by  $\mathrm{Gr}_{+-}(W)$ ) the part of the Grassmannian  $\mathrm{Gr}_{\mathbb{R}}(2, V)$  of 2-dimensional  $\mathbb{R}$ -subspaces in  $V$  formed by the subspaces of signature  $++$  (respectively,  $+-$ ) in  $W$ .

**Definition 3.1:** We shall call a discrete, additive subgroup  $L \subset V$  a **lattice** if  $V = \mathbb{R} \otimes_{\mathbb{Z}} L$  and  $\langle l_1, l_2 \rangle \in \mathbb{Q}$  for all  $l_1, l_2 \in L$ . Denote by  $O(V)$  and  $O(L)$  the corresponding orthogonal groups:

$$\begin{aligned} O(V) &:= \left\{ g \in \mathrm{GL}(V) \mid \langle g(v_1), g(v_2) \rangle = \langle v_1, v_2 \rangle \text{ for all } v_1, v_2 \in V \right\}, \\ O(L) &:= \{ g \in O(V) \mid g(L) = L \}. \end{aligned}$$

Clearly,  $O(V)$  acts on  $\mathrm{Gr}_{++}(V)$ . For  $S \subset V$ , we denote

$$S^\perp := \{ v \in V \mid \langle v, S \rangle = 0 \}.$$

The purpose of the present section is to prove

**Proposition 3.2:** *Let  $V$  be an  $\mathbb{R}$ -vector space equipped with a non-degenerate symmetric form of signature  $(s_+, s_-)$  with  $s_+ \geq 3$  and  $s_- \geq 1$ . Consider a lattice  $L \subset V$ . Let  $\Gamma$  be a subgroup of finite index in  $O(L)$ , and  $l \in L$  a **positive vector**, i.e., one which satisfies  $\langle l, l \rangle > 0$ . Then  $\Gamma \cdot \mathrm{Gr}_{++}(l^\perp)$  is dense in  $\mathrm{Gr}_{++}(V)$ .*

The proof of Proposition 3.2 takes the rest of this Section.

**Proof of Proposition 3.2: Step 1:** We reduce Proposition 3.2 to a case of a space  $V$  of signature  $(3, 1)$ .

A subspace  $W \subset V$  is called **rational** if  $\mathrm{rk}(W \cap L) = \dim_{\mathbb{R}} W$  or, equivalently, if  $W = \mathbb{R}W_0$  with a  $\mathbb{Q}$ -subspace  $W_0 \subset \mathbb{Q}L$ . Since the rational subspaces are dense in  $\mathrm{Gr}_{++}(V)$ , it suffices to show that an arbitrary rational 2-plane  $C \in \mathrm{Gr}_{++}(V)$  belongs to the closure of  $\Gamma \cdot \mathrm{Gr}_{++}(l^\perp)$ . We have  $C = \mathbb{R}C_0$  for some  $\mathbb{Q}$ -subspace  $C_0 \subset \mathbb{Q}L$ .

Obviously,  $\mathbb{Q}L$  has signature  $(s_+, s_-)$ . Applying to  $\mathbb{Q}$ -subspaces in  $\mathbb{Q}L$  the standard orthogonalization arguments, we can find a  $\mathbb{Q}$ -subspace  $U_0 \subset \mathbb{Q}L$  of signature  $+++ -$  that contains both  $l$  and  $C_0$ . Indeed, we have  $l = c_0 + c_1$ , where  $c_0 \in C_0$  and  $c_1 \in \mathbb{Q}L \cap C_0^\perp$  with  $\mathbb{Q}L \cap C_0^\perp$  of signature  $(s_+ - 2, s_-)$ . We can always pick a 2-dimensional  $\mathbb{Q}$ -subspace  $C_1 \subset \mathbb{Q}L \cap C_0^\perp$  of signature  $+ -$  that includes  $c_1$  and put  $U_0 := C_0 \oplus C_1$ . So,  $U := \mathbb{R}U_0$  is rational of signature  $+++ -$  and  $L_0 := U \cap L$  is a lattice in  $U$ . To prove that the 2-plane  $C$  belongs to the closure of  $\Gamma \cdot \mathrm{Gr}_{++}(l^\perp)$ , it would suffice to show that the set  $(\Gamma \cap \Gamma') \cdot \mathrm{Gr}_{++}(l^\perp \cap U)$  is dense in the corresponding  $++$ -Grassmannian  $\mathrm{Gr}_{++}(U)$ , where  $\Gamma' := O(L_0)$ .

**Step 2:** We prove that the orthogonal groups  $O(L)$  and  $O(L')$  are **commensurable**, i.e., the subgroup  $O(L) \cap O(L')$  has finite index in  $O(L)$  and in  $O(L')$ , if lattices  $L, L' \subset V$  are commensurable.

Taking  $L \cap L'$  for  $L'$ , we can assume that  $mL \subset L' \subset L$  for some  $0 \neq m \in \mathbb{Z}$ . Put  $\overline{L'} := L'/mL \subset \overline{L} := L/mL$  and note that  $O(L)$  acts on  $\overline{L}$  because  $O(L) = O(mL)$ . We can see that the group  $O(L) \cap O(L') = \{g \in O(L) \mid g(L') = L'\}$  coincides with the stabilizer  $\text{St}_{O(L)} \overline{L'}$ . Hence,  $O(L) \cap O(L')$  has finite index in  $O(L)$ . Since  $m(\frac{1}{m}L') \subset L \subset \frac{1}{m}L'$  and  $O(\frac{1}{m}L') = O(L')$ , we infer as well that  $O(L) \cap O(L')$  has finite index in  $O(L')$ .

**Step 3:** Let  $W \subset V$  be a rational non-degenerate subspace. Then we have an orthogonal decomposition  $V = W \oplus W^\perp$  and  $W^\perp$  is rational. Define  $L_0 := W \cap L$ ,  $L_1 := W^\perp \cap L$ , and  $L' := L_0 + L_1$ . It is immediate that  $L'$  is a lattice in  $V$  such that  $\mathbb{Q}L' = \mathbb{Q}L$ . By Step 2, the orthogonal groups  $O(L)$  and  $O(L')$  are commensurable. Since  $O(L_0) \times O(L_1) \subset O(L')$ , there exists a subgroup  $\Gamma_0$  of finite index in  $O(L_0)$  such that  $\Gamma_0 \subset \Gamma$ .

**Step 4:** We reduce Proposition 3.2 to Lemma 3.3 below.

Applying Steps 1 and 3, we can assume that  $(s_+, s_-) = (3, 1)$ . Indeed, by Step 1, we need only to show that  $(\Gamma \cap \Gamma') \cdot \text{Gr}_{++}(l^\perp \cap U)$  is dense in  $\text{Gr}_{++}(U)$ , where  $\Gamma' := O(L_0)$ ,  $L_0 := U \cap L$ , and  $U \subset V$  is a rational subspace of signature  $+++$ . Taking  $W := U$  in Step 3, we find a subgroup  $\Gamma_0$  of finite index in  $\Gamma'$  such that  $\Gamma_0 \subset \Gamma$ .

Now using the homeomorphism  $\text{Gr}_{++}(V) \rightarrow \text{Gr}_{+-}(V)$ ,  $G \mapsto G^\perp$ , i.e., taking instead of subspaces of signature  $++$ , their orthogonal complements (of signature  $+-$ ), we reformulate Proposition 3.2 as follows:

*Every rational  $G_0 \in \text{Gr}_{+-}(V)$  belongs to the closure of  $\Gamma \cdot \{G \in \text{Gr}_{+-}(V) \mid G \ni l\}$ .*

The subspace  $W$  spanned by  $l, G_0$  is rational of signature  $++-$ . Again using Step 3, we reduce Proposition 3.2 to

**Lemma 3.3:** *Let  $V$  be an  $\mathbb{R}$ -vector space equipped with a symmetric form of signature  $++-$ ,  $\Gamma$  a subgroup of finite index in  $O(L)$ , where  $L$  is a lattice in  $V$ , and  $l \in V$  a positive vector. Then  $\Gamma \cdot \{G \in \text{Gr}_{+-}(V) \mid G \ni l\}$  is dense in  $\text{Gr}_{+-}(V)$ .*

Till the end of this Section we fix  $\Gamma$  as in Lemma 3.3.

In fact, we deal now with a hyperbolic plane  $\overline{\mathbb{H}}_{\mathbb{R}}^2 = \mathbb{H}_{\mathbb{R}}^2 \sqcup \partial\mathbb{H}_{\mathbb{R}}^2$ . Let us state in Claim 3.4, Claim 3.5, and Claim 3.6 a few simple and well-known facts concerning the hyperbolic plane (see e.g. [AGr]).

**Claim 3.4:** The plane  $\overline{\mathbb{H}}_{\mathbb{R}}^2$  can be identified with the set of all nonpositive points in the real projective plane  $\mathbb{P}_{\mathbb{R}}V$ , where the isotropic ones form the **absolute**  $\partial\mathbb{H}_{\mathbb{R}}^2$ . In the affine chart related to orthonormal coordinates on  $V$ , the plane  $\overline{\mathbb{H}}_{\mathbb{R}}^2$  is nothing but a closed unitary disc. In this way, we obtain the Beltrami-Klein model of a hyperbolic plane, where geodesics are chords of the disc. In other words, we can describe a geodesic in  $\overline{\mathbb{H}}_{\mathbb{R}}^2$  as the projectivization  $\mathbb{P}_{\mathbb{R}}G \cap \overline{\mathbb{H}}_{\mathbb{R}}^2$  of a subspace  $G \in \text{Gr}_{+-}(V)$ . We keep denoting this geodesic by  $G$ . Of course, every geodesic  $G$  can be described via its **vertices**  $v, v' \in \partial\mathbb{H}_{\mathbb{R}}^2$  as  $G = [v, v']$ . In terms of  $V$ , this means that the  $\mathbb{R}$ -vector subspace  $G$  is spanned by  $v, v'$ . ■

**Claim 3.5:** Let  $G' \subset \overline{\mathbb{H}}_{\mathbb{R}}^2$  be a geodesic not passing through a point  $v \in \partial\mathbb{H}_{\mathbb{R}}^2$ , i.e.,  $v \notin G'$ . Then, reflecting  $v$  in  $G'$ , we obtain a point  $v' \in \partial\mathbb{H}_{\mathbb{R}}^2$  such that the geodesics  $G'$  and  $[v, v']$  are orthogonal. ■

**Claim 3.6:** The group  $O(V)$  acts naturally on  $\overline{\mathbb{H}}_{\mathbb{R}}^2$ . On  $\mathbb{H}_{\mathbb{R}}^2$ , the group  $O(V)$  acts by isometries. ■

We can now reduce Lemma 3.3 to the following statement about the hyperbolic plane:

**Lemma 3.7:** Let  $G'$  be a geodesic on the hyperbolic plane  $\mathbb{H}_{\mathbb{R}}^2$ , and  $\Gamma \cdot G'$  the set of all geodesics obtained from  $G'$  by the action of  $\Gamma$ . Then the set of all geodesics orthogonal to some  $G'' \in \Gamma \cdot G'$  is dense in the set of all geodesics in  $\mathbb{H}_{\mathbb{R}}^2$ .

**Reduction of Lemma 3.3 to Lemma 3.7.** Let  $G'$  be the orthogonal complement of  $l \in \mathbb{H}_{\mathbb{R}}^2 \subset \mathbb{P}V$ , considered as a geodesic in  $\mathbb{H}_{\mathbb{R}}^2$ . It is easy to see that the inclusion  $G \ni l$  is equivalent to the fact that the geodesics  $G$  and  $G' := l^{\perp}$  are orthogonal (see, for instance, the duality described in the introductory [AGr, Section 1] shortly after Example 1.7). For this choice of  $G'$ , Lemma 3.3 is clearly equivalent to Lemma 3.7. ■

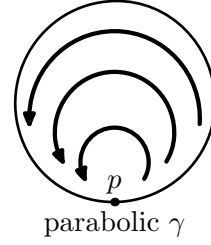
We reduce Lemma 3.7 further, obtaining a simpler statement about the hyperbolic plane:

**Lemma 3.8:** Let  $v, v' \in \partial\mathbb{H}_{\mathbb{R}}^2$  be distinct points on the absolute and  $G'$  a geodesic. For every  $\gamma \in \Gamma$ , denote by  $R_{\gamma}$  the reflection in the geodesic  $\gamma(G')$ . Then  $v'$  belongs to the closure of the set  $\{R_{\gamma}(v) \mid \gamma \in \Gamma, v \notin \gamma(G')\}$  formed by the reflections of  $v$  in those geodesics  $\gamma(G')$  that do not pass through  $v$ .

**Reduction of Lemma 3.7 to Lemma 3.8:** By Claim 3.5, the geodesic  $[v, R_\gamma(v)]$  is orthogonal to  $\gamma(G')$ . To prove Lemma 3.7, it suffices to show that the set of such geodesics is dense in the set of all geodesics of the form  $[v, v']$ , where  $v$  is fixed. Lemma 3.8 says that we are able to approximate  $v'$  by  $R_\gamma(v)$  for an appropriate  $\gamma \in \Gamma$ . Hence, we can approximate the geodesic  $[v, v']$  by geodesics orthogonal to some  $G'' \in \Gamma \cdot G'$ . ■

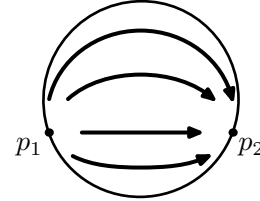
We deduce Lemma 3.8 from two easy lemmas below, Lemma 3.10 and Lemma 3.11. First, we need a few more simple and well-known facts concerning the hyperbolic plane:

- The nontrivial orientation-preserving isometries of  $\mathbb{H}_{\mathbb{R}}^2$  are classified with respect to the location of their fixed points: an **elliptic** one has a (unique) fixed point in  $\mathbb{H}_{\mathbb{R}}^2$ ; a **hyperbolic** one has exactly two fixed points on the absolute; and a **parabolic** one has exactly one fixed point on the absolute.

parabolic  $\gamma$ 

- Let  $p \in \partial\mathbb{H}_{\mathbb{R}}^2$  be the fixed point of a parabolic isometry  $\gamma$  and let  $v \in \partial\mathbb{H}_{\mathbb{R}}^2$ . Then  $\gamma^n(v) \rightarrow p$  as  $n \rightarrow \infty$ .

- The fixed points in  $\partial\mathbb{H}_{\mathbb{R}}^2$  of a hyperbolic isometry  $\gamma$  are the **repeller**  $p_1$  and the **attractor**  $p_2$ . This means that, for every  $v \in \partial\mathbb{H}_{\mathbb{R}}^2$  such that  $v \neq p_1$ , we have  $\gamma^n(v) \rightarrow p_2$  as  $n \rightarrow \infty$ . When taking  $\gamma^{-1}$  in place of  $\gamma$ , the repeller becomes the attractor and vice versa.

hyperbolic  $\gamma$ 

We arrive at the following remark needed in the proof of Lemma 3.8.

**Remark 3.9:** Let  $\gamma$  be a hyperbolic or parabolic isometry,  $p \in \partial\mathbb{H}_{\mathbb{R}}^2$  a fixed point of  $\gamma$ , and  $u, u' \in \partial\mathbb{H}_{\mathbb{R}}^2$  points not fixed by  $\gamma$ . Then, for  $n \rightarrow \infty$  or for  $n \rightarrow -\infty$ , both limits  $\lim \gamma^n(u)$  and  $\lim \gamma^n(u')$  exist and are equal to  $p$ . ■

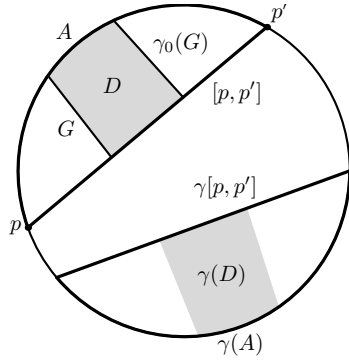
**Lemma 3.10:** The set  $F := \{p \in \partial\mathbb{H}_{\mathbb{R}}^2 \mid \gamma(p) = p \text{ for some } 1 \neq \gamma \in \Gamma\}$  of points on the absolute fixed by some nontrivial  $\gamma \in \Gamma$  is dense in  $\partial\mathbb{H}_{\mathbb{R}}^2$ .

**Proof:** Suppose that there exists an open arc  $A \subset \partial\mathbb{H}_{\mathbb{R}}^2$  such that  $A \cap F = \emptyset$ . By the Zorn lemma, we can take maximal  $A$  with this property. By construction,  $\gamma(A)$  also enjoys the property of the maximality for every  $\gamma \in \Gamma$ . Every point on the boundary  $\partial A$  belongs to the closure of  $F$ . Let

$\gamma \in \Gamma$ . Then  $A \cap \gamma(A) = \emptyset$  or  $A = \gamma(A)$  because otherwise  $\gamma(A)$  contains an open neighbourhood of one end of  $A$ , which intersects  $F$ .

Due to B. A. Venkov (see [VGSh, Example 7.5, p. 33]),  $O(L)$  is known to act discretely on  $\mathbb{H}_{\mathbb{R}}^2$ , is finitely generated, and is of finite coarea. Note that Selberg's Theorem [VGSh, Theorem 3.2, p. 18] claims that every finitely generated matrix group over a field of characteristic 0 has a subgroup of finite index without torsion. Therefore, we can at the very beginning pass to a torsion-free subgroup of finite index in  $\Gamma$  thus assuming that all isometries in  $\Gamma$  are orientation-preserving and that there are no elliptic isometries in  $\Gamma$ .

Let  $\partial A = \{p, p'\}$ . Since  $\Gamma$  has no elliptic isometries and all isometries in  $\Gamma$  are orientation-preserving, the stabilizer  $\Gamma'' := \text{St}_{\Gamma} A$  of  $A$  in  $\Gamma$  is a discrete



group of orientation-preserving isometries of the geodesic  $[p, p']$ . Hence,  $\Gamma''$  is cyclic, generated by some  $\gamma_0 \neq 1$ . Let  $G$  be a geodesic perpendicular to  $[p, p']$ . Consider the open region  $D \subset \mathbb{H}_{\mathbb{R}}^2$  limited by  $A \cup G \cup [p, p'] \cup \gamma_0(G)$ . It is easy to see that  $D \cap \gamma(D) = \emptyset$  for every  $1 \neq \gamma \in \Gamma''$ . For any  $\gamma \in \Gamma \setminus \Gamma''$ , we have  $A \cap \gamma(A) = \emptyset$ , which again implies  $D \cap \gamma(D) = \emptyset$ . Therefore,  $D$  is a part of a fundamental domain for  $\Gamma$ . Since the area of  $D$  is infinite, we arrive at a contradiction. ■

**Lemma 3.11:** *Let  $u, u' \in \partial\mathbb{H}_{\mathbb{R}}^2$  be distinct points. Then there exists a hyperbolic or parabolic  $\gamma_0 \in \Gamma$  such that  $\gamma_0(u) \neq u$  and  $\gamma_0(u') \neq u'$ .*

**Proof:** As in the proof of Lemma 3.10, we assume  $\Gamma$  torsion-free. Suppose that  $\gamma_0(u) = u$  or  $\gamma_0(u') = u'$  for every  $\gamma_0 \in \Gamma$ . If  $\gamma, \gamma' \in \Gamma$  fix respectively  $u, u'$  and do not fix respectively  $u', u$ , then  $\gamma\gamma'$  does not fix both  $u$  and  $u'$ . Therefore, we can assume that  $\gamma(u) = u$  for all  $\gamma \in \Gamma$ . It is well known (consider the upper half-plane model with  $u = 0$ ) that the group of all orientation-preserving isometries of  $\mathbb{H}_{\mathbb{R}}^2$  is isomorphic to  $\text{PSL}_2(\mathbb{R})$  and that  $S := \text{St}_{\text{PSL}_2(\mathbb{R})} u \simeq \left\{ \begin{bmatrix} \alpha & 0 \\ a & \alpha^{-1} \end{bmatrix} \mid a, \alpha \in \mathbb{R}, \alpha > 0 \right\}$ . Since  $S$  is Zariski closed, the inclusion  $\Gamma \subset S$  would contradict the Borel density theorem [VGSh, Theorem 8.2, p. 37] which implies that  $\Gamma$  should be Zariski dense in  $\text{PSL}_2(\mathbb{R})$ . ■

**Proof of Lemma 3.8:** For suitable distinct points  $u, u' \in \partial\mathbb{H}_{\mathbb{R}}^2$ , the geodesic  $G'$  in Lemma 3.8 has the form  $G' = [u, u']$ .

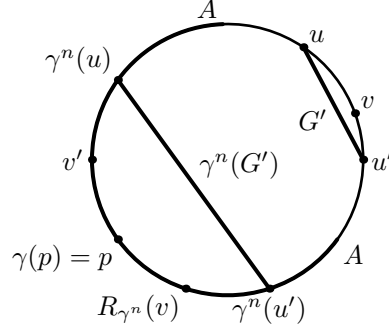
Let  $A$  be a small connected open neighbourhood of  $v'$  in  $\partial\mathbb{H}_{\mathbb{R}}^2$ . In other words,  $A \subset \partial\mathbb{H}_{\mathbb{R}}^2$  is an open arc containing  $v'$  and not containing  $v$ . By

Lemma 3.10, for a suitable  $p \in A \cap F$  and for some  $1 \neq \gamma \in \Gamma$ , we have  $\gamma(p) = p$ .

We consider two cases. The first case:  $u, u'$  are not fixed by  $\gamma$ . Then, taking into account that  $\gamma$  is hyperbolic or parabolic, we conclude by Remark 3.9 that  $\gamma^n(u) \rightarrow p$  and  $\gamma^n(u') \rightarrow p$  for  $n \rightarrow \infty$  or for  $n \rightarrow -\infty$ . Hence,  $\gamma^n(u), \gamma^n(u') \in A$  for some  $n \in \mathbb{Z}$ . Therefore,  $R_{\gamma^n}(v) \in A$ .

The second case: one of  $u, u'$  is fixed by  $\gamma$ . By Lemma 3.11, there exists  $\gamma_0 \in \Gamma$  such that the points  $\gamma_0(u), \gamma_0(u')$  are not fixed by  $\gamma$ . Now, by Remark 3.9, we have  $\gamma^n \gamma_0(u) \rightarrow p$  and  $\gamma^n \gamma_0(u') \rightarrow p$  for  $n \rightarrow \infty$  or for  $n \rightarrow -\infty$ . This implies  $R_{\gamma^n \gamma_0}(v) \in A$  for some  $n \in \mathbb{Z}$ .

For an arbitrarily small open arc  $A$  containing  $v'$ , we found, in either case, some  $\gamma' \in \Gamma$  such that  $R_{\gamma'}(v) \in A$  and  $v \notin \gamma'(G')$ . This implies Lemma 3.8. ■



**Remark 3.12:** We stated Proposition 3.2 in assumption that  $\langle l, l \rangle > 0$  (this assumption was geometrically motivated). But, in fact, this assumption is completely unnecessary. Moreover, as the following result implies, the proof of Proposition 3.2 becomes much easier when  $\langle l, l \rangle \leq 0$ .

**Proposition 3.13:** The condition  $\langle l, l \rangle > 0$  in Proposition 3.2 is unnecessary.

**Proof:** To see this, we repeat the proof of Proposition 3.2 literally until Lemma 3.3. To obtain Remark 3.12, we need to check a version of Lemma 3.3 when the vector  $l$  is not assumed to be positive.

We reduce the case of  $\langle l, l \rangle < 0$  to the case of  $\langle l, l \rangle = 0$ . Let  $l_0$  be a limit point of the orbit  $\Gamma \cdot l$ . Since  $\Gamma$  is a discrete subgroup in  $\mathrm{PSL}_2(\mathbb{R})$ , this limit lies on the absolute, and we have  $\langle l_0, l_0 \rangle = 0$ . It suffices to show that any geodesic  $G$  passing through  $l_0$  lies in the closure of the set of all geodesics  $G'$  that pass through  $\gamma(l)$  for some  $\gamma \in \Gamma$ . For a given point  $\gamma(l)$ , we denote by  $G'_\gamma$  the Euclidean parallel to  $G$  passing through  $\gamma(l)$ . For this choice of  $G'_\gamma$ , the limit  $\gamma(l) \rightarrow l_0$  implies the limit  $G'_\gamma \rightarrow G$ .

It remains now to prove Lemma 3.3 when  $\langle l, l \rangle = 0$ . Since  $F$  is dense in  $\partial \mathbb{H}_{\mathbb{R}}^2$  by Lemma 3.10, the subset  $\Gamma \cdot l$  is also dense in  $\partial \mathbb{H}_{\mathbb{R}}^2$ . So, fixing one end of an arbitrary geodesic  $G \in \mathrm{Gr}_{+-}(V)$ , we can approximate the other one by a point in  $\Gamma \cdot l$ . ■

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MISHA VERBITSKY

LABORATORY OF ALGEBRAIC GEOMETRY, NRU-HSE,  
7 VAVILOVA STR. MOSCOW, RUSSIA, 117312  
`verbit@maths.gla.ac.uk`, `verbit@mccme.ru`

SASHA ANAN'IN

IMECC – UNICAMP, DEPARTAMENTO DE MATEMÁTICA,  
CAIXA POSTAL 6065 13083-970 CAMPINAS-SP, BRAZIL  
`ananin_sasha@yahoo.com`

